

Weak Maximum Principle for Strongly Coupled Elliptic Differential Systems

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Abstract

A classical counterexample due to E. De Giorgi, shows that the weak maximum principle does not remain true for general linear elliptic differential systems. After that, there are some efforts to establish the weak maximum principle for special elliptic differential systems, but the existing works are addressing only the cases of weakly coupled systems, or almost-diagonal systems, or even some systems coupling in various lower order terms. In this paper, by contrast, we present maximum modulus estimates for weak solutions to two classes of coupled linear elliptic differential systems with different principal parts, under considerably mild and physically reasonable assumptions. The systems under consideration are strongly coupled in the second order terms and other lower order terms, without restrictions on the size of ratios of the different principal part coefficients, or on the number of equations and space variables.

Key Words. Weak maximum principle, strongly coupled elliptic system, weak solution

1 Introduction

Let $m, n \in \mathbb{N} \setminus \{0\}$, and $\Omega \subset \mathbb{R}^m$ be a bounded domain with an C^1 boundary Γ and having the cone property. We consider the following nonhomogeneous, isotropic elliptic differential system of second order:

$$\left\{ \begin{array}{ll} -\operatorname{div}(a^{11}\nabla y^1) - \operatorname{div}(a^{12}\nabla y^2) - \cdots - \operatorname{div}(a^{1n}\nabla y^n) + \sum_{i=1}^n C^{1i} \cdot \nabla y^i + D^1 \cdot y = f^1 & \text{in } \Omega, \\ -\operatorname{div}(a^{21}\nabla y^1) - \operatorname{div}(a^{22}\nabla y^2) - \cdots - \operatorname{div}(a^{2n}\nabla y^n) + \sum_{i=1}^n C^{2i} \cdot \nabla y^i + D^2 \cdot y = f^2 & \text{in } \Omega, \\ \vdots & \\ -\operatorname{div}(a^{n1}\nabla y^1) - \operatorname{div}(a^{n2}\nabla y^2) - \cdots - \operatorname{div}(a^{nn}\nabla y^n) + \sum_{i=1}^n C^{ni} \cdot \nabla y^i + D^n \cdot y = f^n & \text{in } \Omega, \\ y^1 = g^1, \quad y^2 = g^2, \quad \cdots, \quad y^n = g^n & \text{on } \Gamma, \end{array} \right. \quad (1.1)$$

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and the following general nonhomogeneous elliptic differential system of second order:

$$\left\{ \begin{array}{ll} - \sum_{p,q=1}^m \left[(a_{pq}^{11} y_{x_p}^1)_{x_q} + (a_{pq}^{12} y_{x_p}^2)_{x_q} + \cdots + (a_{pq}^{1n} y_{x_p}^n)_{x_q} \right] + \sum_{i=1}^n C^{1i} \cdot \nabla y^i + D^1 \cdot y = f^1 & \text{in } \Omega, \\ - \sum_{p,q=1}^m \left[(a_{pq}^{21} y_{x_p}^1)_{x_q} + (a_{pq}^{22} y_{x_p}^2)_{x_q} + \cdots + (a_{pq}^{2n} y_{x_p}^n)_{x_q} \right] + \sum_{i=1}^n C^{2i} \cdot \nabla y^i + D^2 \cdot y = f^2 & \text{in } \Omega, \\ \vdots & \\ - \sum_{p,q=1}^m \left[(a_{pq}^{n1} y_{x_p}^1)_{x_q} + (a_{pq}^{n2} y_{x_p}^2)_{x_q} + \cdots + (a_{pq}^{nn} y_{x_p}^n)_{x_q} \right] + \sum_{i=1}^n C^{ni} \cdot \nabla y^i + D^n \cdot y = f^n & \text{in } \Omega, \\ y^1 = g^1, \quad y^2 = g^2, \quad \cdots, \quad y^n = g^n & \text{on } \Gamma. \end{array} \right. \quad (1.2)$$

In both (1.1) and (1.2), $y = (y^1, \dots, y^n)^\top$ is unknown, while a^{ij} , a_{pq}^{ij} , C^{ij} , D^i , f^i and g^i ($i, j = 1, \dots, n$; $p, q = 1, \dots, m$) are suitable given functions (See the next section for the assumptions on these functions). The main purpose of this paper is to study the weak maximum principle, or the boundedness of weak solutions, for systems (1.1) and (1.2) with suitable measurable principal part coefficients.

It is well-known that the weak maximum principle is one of the basic issues in the theory of partial differential equations and it plays an essential role in the study of many other problems. For example, a central problem in the calculus of variations is the regularity of stationary points for functionals of the type

$$J(u) = \int_{\Omega} F(x, u(x), \nabla u(x)) dx,$$

where $u(x) = (u^1(x), u^2(x), \dots, u^n(x))^\top$ is a vector-valued function defined on Ω , and $F(\cdot, \cdot, \cdot)$ is a suitable function defined on $\Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$. Research in this area has been stimulated by D. Hilbert's Problem 19, which can be reduced to the regularity of weak solutions to nonlinear elliptic equations or systems. This problem was successfully solved by C. Morrey ([14]) in two dimensions and the general case with $n = 1$ was finally solved by E. De Giorgi ([7]) and J. Nash ([16]), and refined by J. Moser ([15]). We refer to [1, 2, 9, 10, 11] and the references cited therein for more details in this respect. A fundamental step of the De Giorgi-Nash-Moser approach in solving the scalar Hilbert's 19th problem (i.e., $n = 1$) is to establish the weak maximum principle for single linear elliptic equations.

In many physical and geometrical applications, u may be a vector function, and therefore, the corresponding Euler-Lagrange equation is a system. Naturally, one expects to extend the De Giorgi-Nash-Moser approach to the case of systems. However, in 1968, E. De Giorgi ([8]) gave a surprising counterexample of an unbounded solution to a second order linear elliptic system with bounded coefficients. This means that the weak maximum principle fails for general second order linear elliptic systems, and therefore, the De Giorgi-Nash-Moser estimates are no longer valid for general elliptic systems.

In order to establish the weak maximum principle for elliptic differential systems, as a consequence of the above mentioned De Giorgi's counterexample, one has to impose some restrictions on the structure of the system. There exist a few works in this direction. In [11], a weak maximum principle was proved for a class of special elliptic systems with variable coefficients, in which the

principal operator in each equation takes the same form, and it is acting only on one component of the solution vector. In [3, 4, 12], some weak maximum principles were discussed in the frame of Campanato's space for linear or quasilinear elliptic systems under some additional conditions, say, $2 \leq m \leq 4$ in [3], the coefficients matrix being constant in [4], and a dispersion assumption on the eigenvalues of the principal part coefficients matrix in [12] (and hence the system is almost-diagonal in high space dimensions).

In this paper, we choose the usual Sobolev space as the working space and derive weak maximum principles for two classes of strongly coupled elliptic systems with different principal parts, in the spirit of the classical framework for single equations. We emphasize that our systems are strongly coupled, i.e., the (second order) terms of the principal parts are coupled each other. Therefore, when establishing the desired *a priori* estimate, it is necessary to get rid of some undesired terms generated by different principal operators and/or different solution components appeared in the same equation. This goal is achieved by choosing delicately suitable weighted test functions. As far as we know, this is the first result on the weak maximum principle (in the classical sense) for strongly coupled elliptic systems.

The rest of this paper is organized as follows. Section 2 is devoted to stating the main results in this work. In Section 3, we collect some preliminary results which will be useful later. Sections 4 and 5 are addressed to the proof of the main results, i.e., the boundedness of weak solutions to systems (1.1) and (1.2), respectively. Finally, in Section 6, we give an example in which the assumptions for proving the boundedness of the weak solution to system (1.2) are satisfied.

2 Statement of the main results

To begin with, we introduce some assumptions. Suppose that, for $i, j = 1, \dots, n$,

$$a^{ij} \in L^\infty(\Omega) \quad (2.1)$$

and

$$\begin{cases} C^{ij}(\cdot) \in L^\theta(\Omega; \mathbb{R}^m) \text{ and } D^i(\cdot) \in L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n) \text{ for some } \theta > m, \\ f = (f^1, \dots, f^n)^\top \in H^{-1}(\Omega; \mathbb{R}^n), \quad g = (g^1, \dots, g^n)^\top \in H^1(\Omega; \mathbb{R}^n), \end{cases} \quad (2.2)$$

and for $i, j = 1, \dots, n$ and $p, q = 1, \dots, m$,

$$a_{pq}^{ij} \in L^\infty(\Omega), \quad a_{pq}^{ij} = a_{qp}^{ij}. \quad (2.3)$$

Moreover, we assume that, for some positive constant ρ ,

$$\sum_{i,j=1}^n \sum_{p=1}^m a^{ij} \xi_p^i \xi_p^j \geq \rho |\xi|^2, \quad (2.4)$$

$$\forall (x, \xi) = (x, \xi_1^1, \dots, \xi_m^1, \dots, \xi_1^n, \dots, \xi_m^n) \in \Omega \times \mathbb{R}^{nm},$$

and

$$\sum_{i,j=1}^n \sum_{p,q=1}^m a_{pq}^{ij}(x) \xi_p^i \xi_q^j \geq \rho |\xi|^2, \quad (2.5)$$

$$\forall (x, \xi) = (x, \xi_1^1, \dots, \xi_m^1, \dots, \xi_1^n, \dots, \xi_m^n) \in \Omega \times \mathbb{R}^{nm}.$$

Conditions (2.4) and (2.5) mean that both systems (1.1) and (1.2) are elliptic (see [5, Section 1 of Chapter 8]). Clearly, system (1.1) is a special case of system (1.2). The weak solution to system (1.2) is understood in the following sense:

Definition 2.1 We call $y = (y^1, \dots, y^n)^\top \in H^1(\Omega; \mathbb{R}^n)$ to be a weak solution to system (1.2) if for any $\varphi = (\varphi^1, \dots, \varphi^n)^\top \in H_0^1(\Omega; \mathbb{R}^n)$,

$$\begin{aligned} & \sum_{i,j=1}^n \sum_{p,q=1}^m \int_{\Omega} a_{pq}^{ij}(x) y_{x_p}^j \varphi_{x_q}^i dx + \int_{\Omega} \sum_{i=1}^n \left[\sum_{j=1}^n C^{ij}(x) \cdot \nabla y^j \varphi^i + D^i(x) \cdot y \varphi^i \right] dx \\ &= \langle f, \varphi \rangle_{H^{-1}(\Omega; \mathbb{R}^n), H_0^1(\Omega; \mathbb{R}^n)}, \end{aligned}$$

and $y^i - g^i \in H_0^1(\Omega)$, $i = 1, \dots, n$.

Similar to the proof of [5, Theorem 2.3 in Chapter 1]), it is easy to show the following well-posedness result for system (1.2).

Lemma 2.1 Let conditions (2.2), (2.3) and (2.5) be fulfilled. Then, there exists a constant $\nu_0 = \nu_0(n, m, \theta) > 0$ such that system (1.2) admits a unique weak solution $y \in H^1(\Omega; \mathbb{R}^n)$ whenever the following inequality

$$\begin{aligned} \sum_{i=1}^n (D^i(x) \cdot \mu) \mu^i &\geq \nu \rho^{\frac{m+\theta}{m-\theta}} \left[\sum_{i,j=1}^n |C^{ij}|_{L^\theta(\Omega; \mathbb{R}^m)} \right]^{\frac{2\theta}{\theta-m}} |\mu|^2, \\ \forall (x, \mu) &= (x, \mu^1, \mu^2, \dots, \mu^n) \in \Omega \times \mathbb{R}^n \end{aligned} \quad (2.6)$$

holds for $\nu \geq \nu_0$. Moreover,

$$|y|_{H^1(\Omega; \mathbb{R}^n)} \leq C \left(n, m, \Omega, \rho, |a_{pq}^{ij}|_{L^\infty(\Omega)}, |C^{ij}|_{L^\theta(\Omega; \mathbb{R}^m)}, |D^i|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)} \right) (|f|_{H^{-1}(\Omega; \mathbb{R}^n)} + |g|_{H^1(\Omega; \mathbb{R}^n)}).$$

The proof of Lemma 2.1 is standard and therefore we omit the details.

Remark 2.1 Clearly, if $C^{ij}(\cdot) \equiv 0$ for $i, j = 1, \dots, n$, then condition (2.6) is satisfied whenever the function matrix $(D^1(x), \dots, D^n(x))$ is semi-positive definite.

Next, we put

$$A = \begin{pmatrix} a^{11} & a^{21} & \dots & a^{n1} \\ a^{12} & a^{22} & \dots & a^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a^{1n} & a^{2n} & \dots & a^{nn} \end{pmatrix}, \quad B = \begin{pmatrix} a^{22} & a^{32} & \dots & a^{n2} \\ a^{23} & a^{33} & \dots & a^{n3} \\ \vdots & \vdots & \ddots & \vdots \\ a^{2n} & a^{3n} & \dots & a^{nn} \end{pmatrix}.$$

Also, denote by B^{ij} ($i, j = 1, \dots, n$) the cofactor of A with respect to a^{ij} and by $\det A$ the determinant of matrix A . It is easy to see that $B^{11} = B$. Moreover, under condition (2.4), it is easy to show that $\det B \neq 0$.

The first main result in this paper is the following boundedness of weak solutions to system (1.1).

Theorem 2.1 Suppose that conditions (2.1), (2.2) and (2.4) are fulfilled, inequality (2.6) holds for $\nu \geq \nu_0$ (Recall Lemma 2.1 for ν_0), $f \in L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)$ and

$$\frac{\det B^{ij}}{\det B} \in W^{1,\infty}(\Omega), \quad i, j = 1, \dots, n. \quad (2.7)$$

Then the weak solution $y \in H^1(\Omega; \mathbb{R}^n)$ to system (1.1) satisfies

$$\operatorname{esssup}_{\Omega} |y| \leq C \left(m, n, \theta, \Omega, \rho, |a^{ij}|_{L^\infty(\Omega)}, |C^{ij}|_{L^\theta(\Omega; \mathbb{R}^m)}, |D^i|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)}, \left| \frac{\det B^{ij}}{\det B} \right|_{W^{1,\infty}(\Omega)}, \right. \\ \left. |g|_{H^1(\Omega; \mathbb{R}^n)}, |f|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)}, \operatorname{esssup}_{\Gamma} |y| \right).$$

The proof of Theorem 2.1 will be given in Section 4.

Remark 2.2 We conjecture that assumption (2.7) in Theorem 2.1 is a technical condition, and therefore it is not really necessary. However, we do not know how to drop this assumption at this moment.

Since almost all of the natural materials are isotropic, Theorem 2.1 suffices for most of physical applications. Nevertheless, from the mathematical point of view, it would be quite interesting to extend Theorem 2.1 to more general anisotropic systems such as (1.2), in which the scalar functions a^{ij} ($i, j = 1, 2, \dots, n$) appeared in system (1.1) are replaced by the $\mathbb{R}^{m \times m}$ matrix-valued functions $(a_{pq}^{ij})_{1 \leq p, q \leq m}$. Note however that, by the above mentioned De Giorgi's counterexample ([8]), this seems to be highly nontrivial in the general case. In the rest of this section, we shall extend Theorem 2.1 to system (1.2) under some technical assumptions.

In order to treat system (1.2), we put

$$M_{pq} = \begin{pmatrix} a_{pq}^{11} & a_{pq}^{21} & \cdots & a_{pq}^{n1} \\ a_{pq}^{12} & a_{pq}^{22} & \cdots & a_{pq}^{n2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{pq}^{1n} & a_{pq}^{2n} & \cdots & a_{pq}^{nn} \end{pmatrix}, \quad L_{pq} = \det M_{pq}, \quad (p, q = 1, \dots, m).$$

We assume that

$$L_{pq} \neq 0, \quad \forall p, q = 1, \dots, m. \quad (2.8)$$

Also, we denote by v_{pq}^{ij} ($i, j = 1, \dots, n$; $p, q = 1, \dots, m$) the cofactor of M_{pq} with respect to a_{pq}^{ij} .

Further, let us introduce the following assumption:

(H) There exist functions $f_{pq}, h^{ij} \in W^{1,\infty}(\Omega)$ ($i, j = 1, \dots, n$; $p, q = 1, \dots, m$) such that

1) $h^{11} \equiv 1$, $h^{ij} = h^{ji}$, and the following matrix is uniformly positive definite:

$$V := \begin{pmatrix} 1 & h^{12} & \cdots & h^{1n} \\ h^{21} & h^{22} & \cdots & h^{2n} \\ \vdots & \vdots & \vdots & \vdots \\ h^{n1} & h^{n2} & \cdots & h^{nn} \end{pmatrix},$$

i.e., $V \geq \rho_1 I_{n \times n}$ for some positive number ρ_1 ;

2) The function $E^{ij} := \frac{f_{pq}}{L_{pq}} \sum_{l=1}^n h^{lj} v_{pq}^{li}$ is independent of p and q , and $E^{ij} \in W^{1,\infty}(\Omega)$ for any $i, j = 1, \dots, n$;

3) The following matrix is uniformly positive definite:

$$F := \begin{pmatrix} F_{11} & F_{12} & \cdots & F_{1m} \\ F_{21} & F_{22} & \cdots & F_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ F_{m1} & F_{m2} & \cdots & F_{mm} \end{pmatrix},$$

i.e., $F \geq \rho_2 I_{m \times m}$ for some positive number ρ_2 , where $F_{pq} := \sum_{l=1}^n a_{pq}^{l1} E^{l1}$ for any $p, q = 1, \dots, m$;

4) The following matrix is uniformly positive definite:

$$M := \begin{pmatrix} F & h^{12}F & \cdots & h^{1n}F \\ h^{21}F & h^{22}F & \cdots & h^{2n}F \\ \vdots & \vdots & \ddots & \vdots \\ h^{n1}F & h^{n2}F & \cdots & h^{nn}F \end{pmatrix}_{nm \times nm},$$

i.e., $M \geq \rho_3 I_{nm \times nm}$ for some positive number ρ_3 .

Now, we can state our another main result as the following boundedness result for weak solutions to system (1.2).

Theorem 2.2 Suppose that conditions (2.3), (2.5) and (2.8) are fulfilled, $C^{ij}(\cdot) \in L^\infty(\Omega; \mathbb{R}^m)$, $D^i(\cdot) \in L^\infty(\Omega; \mathbb{R}^n)$ and inequality (2.6) holds for $\nu \geq \nu_0$ ((Recall Lemma 2.1 for ν_0)), $f \in L^\infty(\Omega; \mathbb{R}^n)$, and assumption **(H)** holds. Then the weak solution $y \in H^1(\Omega; \mathbb{R}^n)$ to system (1.2) satisfies

$$\operatorname{esssup}_\Omega |y| \leq C(m, n, \Omega, \rho, \rho_1, \rho_2, \rho_3, |a_{pq}^{ij}|_{L^\infty(\Omega)}, |C^{ij}|_{L^\infty(\Omega; \mathbb{R}^m)}, |D^i|_{L^\infty(\Omega; \mathbb{R}^n)}, |E^{ij}|_{W^{1,\infty}(\Omega)}, |h^{ij}|_{W^{1,\infty}(\Omega)}, |g|_{H^1(\Omega; \mathbb{R}^n)}, |f|_{L^\infty(\Omega; \mathbb{R}^n)}, \operatorname{esssup}_\Gamma |y|).$$

The proof of Theorem 2.2 will be given in Section 5. Also, in Section 6, we shall give an illustrative example, in which all of the assumptions in Theorem 2.2 are satisfied.

Remark 2.3 It is well-known that one of the classical topics in partial differential equations is the strong maximum principle for elliptic differential equations, which has many applications ([6, 17, 18, 19] and so on). However, the existing results on strong maximum principle are mainly focusing on single elliptic equations, although one can find some works on weakly coupled elliptic systems ([1, 13, 20]) and the references therein. It would be quite interesting to establish a strong maximum principle for system (1.1) or even for system (1.2), but this remains to be done and it seems to be far from easy.

3 Some preliminaries

In this section, we collect some known preliminary results which will be useful later.

The first one is the following interpolation result.

Lemma 3.1 ([11, Theorem 2.1 in Chapter 2]) *For any $u \in W_0^{1,t}(\Omega)$, $t \geq 1$ and $\tau \geq 1$, it holds that*

$$|u|_{L^{p^*}(\Omega)} \leq \beta |\nabla u|_{L^t(\Omega)}^\alpha |u|_{L^\tau(\Omega)}^{1-\alpha},$$

where $\alpha = \left(\frac{1}{\tau} - \frac{1}{p^*}\right) \left(\frac{1}{\tau} - \frac{1}{t^*}\right)^{-1}$, $t^* = \frac{tm}{m-t}$, and β is a constant depending only on m, t, p^*, τ and α . Moreover, if $t < m$, p^* can be any number between τ and t^* ; if $t \geq m$, p^* can be any number larger than τ .

For any Lebesgue measurable function u defined on Ω , we put $A_k = \{x \in \Omega; u(x) > k\}$ and denote by $|A_k|$ the Lebesgue measure of set A_k . The next lemma is quite useful in deriving the supremum of function u .

Lemma 3.2 ([11, Theorem 5.1 in Chapter 2]) *Suppose that $u \in W^{1,m_0}(\Omega) \cap L^{q_0}(\Omega)$ for some $m_0 \in [1, m]$ and some $q_0 \geq 1$. If for any fixed $k \geq \operatorname{esssup}_\Gamma u$, function u satisfies the following inequality:*

$$\int_{A_k} |\nabla u|^{m_0} dx \leq \gamma \left[\int_{A_k} (u - k)^{l_0} dx \right]^{\frac{m_0}{l_0}} + \gamma k^\sigma |A_k|^{1 - \frac{m_0}{m} + \varepsilon_0},$$

where γ, l_0, σ and ε_0 are positive constants satisfying $l_0 < \frac{mm_0}{m-m_0}$ and $m_0 \leq \sigma < \varepsilon_0 q_0 + m_0$, then

$$\operatorname{esssup}_\Omega u \leq C^*(\Omega, m_0, q_0, \gamma, l_0, \sigma, \varepsilon_0, \operatorname{esssup}_\Gamma u, |u|_{L^{q_0}(\Omega)}).$$

Moreover, when $\sigma = m_0$, $|u|_{L^{q_0}(\Omega)}$ appeared in C^* can be replaced by $|u|_{L^1(\Omega)}$.

The last lemma is a result on comparison of the determinants between a matrix and its symmetrizing matrix.

Lemma 3.3 ([21, Theorem 3.7.1]) *For a real matrix E , if $H(E) = \frac{E + E^\top}{2}$ is positive definite, then*

$$\det H(E) \leq \det E.$$

4 Proof of Theorem 2.1

The goal of this section is to prove our first main result, i.e., Theorem 2.1.

Proof of Theorem 2.1. First, for the weak solution $y = (y^1, y^2, \dots, y^n)^\top$ to system (1.1), any fixed $k \geq \operatorname{esssup}_\Gamma |y|^2$ and $r > 0$, put

$$\phi_r(x) = \min\{(|y(x)|^2 - k)_+, r\},$$

where $s_+ = \max\{s, 0\}$ (for any $s \in \mathbb{R}$). We choose $\varphi = (\varphi^1, \varphi^2, \dots, \varphi^n)^\top$ as a test function, where $\varphi^i = (y^1 T^{1i} + y^2 T^{2i} + \dots + y^n T^{ni})\phi_r$ and T^{ij} ($i, j = 1, 2, \dots, n$) are suitable functions to be specified later. Obviously, $\varphi \in H_0^1(\Omega; \mathbb{R}^n)$. By Definition 2.1, it follows that

$$\begin{aligned} & \sum_{i=1}^n \int_{\Omega} (a^{i1} \nabla y^1 + a^{i2} \nabla y^2 + \dots + a^{in} \nabla y^n) \cdot \nabla [(y^1 T^{1i} + y^2 T^{2i} + \dots + y^n T^{ni})\phi_r] dx \\ & + \sum_{i=1}^n \int_{\Omega} \left(\sum_{j=1}^n C^{ij} \cdot \nabla y^j + D^i \cdot y \right) (y^1 T^{1i} + y^2 T^{2i} + \dots + y^n T^{ni})\phi_r dx \\ & = \sum_{i=1}^n \int_{\Omega} f^i (y^1 T^{1i} + y^2 T^{2i} + \dots + y^n T^{ni})\phi_r dx. \end{aligned}$$

This implies that, for any $\varepsilon_1 > 0$,

$$\begin{aligned} & \sum_{i=1}^n \int_{\Omega} \left\{ (a^{i1} T^{1i} |\nabla y^1|^2 + a^{i2} T^{2i} |\nabla y^2|^2 + \dots + a^{in} T^{ni} |\nabla y^n|^2) \phi_r \right. \\ & + \sum_{l,j \in \{1,2,\dots,n\}, l \neq j} \left(a^{ij} T^{li} \nabla y^j \cdot \nabla y^l \phi_r + a^{ij} T^{li} y^l \nabla y^j \cdot \nabla \phi_r \right) \\ & + \frac{1}{2} [a^{i1} T^{1i} \nabla (y^1)^2 + a^{i2} T^{2i} \nabla (y^2)^2 + \dots + a^{in} T^{ni} \nabla (y^n)^2] \cdot \nabla \phi_r \Big\} dx \\ & = \sum_{i=1}^n \int_{\Omega} \left[f^i (y^1 T^{1i} + y^2 T^{2i} + \dots + y^n T^{ni})\phi_r - \sum_{l,j=1}^n a^{il} y^j \nabla y^l \cdot \nabla T^{ji} \phi_r \right. \\ & \quad \left. - \left(\sum_{j=1}^n C^{ij} \cdot \nabla y^j + D^i \cdot y \right) (y^1 T^{1i} + y^2 T^{2i} + \dots + y^n T^{ni})\phi_r \right] dx \\ & \leq C \int_{\Omega} |f| |y| \phi_r dx + \varepsilon_1 \int_{\Omega} |\nabla y|^2 \phi_r dx + C \int_{\Omega} \left[\varepsilon_1^{-1} \left(\sum_{i,j=1}^n |C^{ij}|^2 + 1 \right) + \sum_{i=1}^n |D^i| \right] |y|^2 \phi_r dx. \end{aligned} \tag{4.1}$$

Here and hereafter C denotes a generic constant (which may be different from one place to another), depending only on $n, m, \rho, |a^{ij}|_{L^\infty(\Omega)}$ and $|T^{ij}|_{W^{1,\infty}(\Omega)}$ ($i, j = 1, 2, \dots, n$). Also, it is clear that

$$\begin{aligned} & [a^{i1} T^{1i} \nabla (y^1)^2 + a^{i2} T^{2i} \nabla (y^2)^2 + \dots + a^{in} T^{ni} \nabla (y^n)^2] \cdot \nabla \phi_r \\ & = a^{i1} T^{1i} \nabla |y|^2 \cdot \nabla \phi_r + [(a^{i2} T^{2i} - a^{i1} T^{1i}) \nabla (y^2)^2 + \dots + (a^{in} T^{ni} - a^{i1} T^{1i}) \nabla (y^n)^2] \cdot \nabla \phi_r. \end{aligned} \tag{4.2}$$

Next, we choose suitable functions T^{ij} ($i, j = 1, 2, \dots, n$) such that $T^{ij} \in W^{1,\infty}(\Omega)$ and some undesired terms in (4.1) vanish. For this purpose, we consider the following linear system:

$$\left\{ \begin{array}{l} \sum_{i=1}^n a^{ij} T^{li} = 0, \quad \forall j, l = 1, 2, \dots, n \text{ with } j \neq l, \\ \sum_{i=1}^n (a^{i2} T^{2i} - a^{i1} T^{1i}) = 0, \\ \vdots \\ \sum_{i=1}^n (a^{in} T^{ni} - a^{i1} T^{1i}) = 0. \end{array} \right. \tag{4.3}$$

By Lemma 3.3 and (2.4), it follows that $\det A \geq \rho^n$ and $\det B \geq \rho^{n-1}$. One can easily check that the following is a solution to system (4.3):

$$\begin{cases} T^{11} = 1, & T^{1i} = (-1)^{1+i} \frac{\det B^{1i}}{\det B} \quad (i = 2, \dots, n), \\ T^{ji} = (-1)^{i+j} \sum_{l=1}^n a^{l1} T^{1l} \frac{\det B^{ji}}{\det A} \quad (j = 2, 3, \dots, n, \quad i = 1, 2, \dots, n). \end{cases} \quad (4.4)$$

Also, it is not difficult to check that

$$\sum_{l=1}^n a^{l1} T^{1l} = \frac{\det A}{\det B}.$$

From this fact and noting (4.4), we see that $T^{ji} = (-1)^{i+j} \frac{\det B^{ji}}{\det B} \quad (i, j = 1, 2, \dots, n)$. Moreover, there exists a constant $\rho^* > 0$, depending only on n, ρ and $|a^{ij}|_{L^\infty(\Omega)} \quad (i, j = 2, \dots, n)$, such that

$$\sum_{l=1}^n a^{l1} T^{1l} = \dots = \sum_{l=1}^n a^{ln} T^{nl} \geq \frac{\rho^n}{\det B} \geq \rho^*. \quad (4.5)$$

Combining (4.2), (4.3) and (4.5) with (4.1), we arrive at

$$\int_{\Omega} (|\nabla y|^2 \phi_r + |\nabla \phi_r|^2) dx \leq C \left[\int_{\Omega} |f| |y| \phi_r dx + \int_{\Omega} \left(\sum_{i,j=1}^n |C^{ij}|^2 + \sum_{i=1}^n |D^i| + 1 \right) |y|^2 \phi_r dx \right]. \quad (4.6)$$

In the sequel, we estimate the terms in the right side of (4.6) and prove that the left side of this inequality is uniformly bounded with respect to $r > 0$. First of all, by Hölder's inequality and Lemma 3.1, we see that

$$\begin{aligned} \int_{\Omega} |f| |y| \phi_r dx &\leq \int_{\Omega} |f| (|y|^2 - k)^{\frac{1}{2}} \phi_r dx + k^{\frac{1}{2}} \int_{\Omega} |f| \phi_r dx \\ &\leq |f|_{L^{\frac{4\theta}{\theta+6}}(\Omega; \mathbb{R}^n)} \left| (|y|^2 - k)^{\frac{1}{2}} \phi_r \right|_{L^{\frac{4\theta}{3\theta-6}}(\Omega)} + C_1 |f|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)} |\phi_r|_{L^{\frac{\theta}{\theta-2}}(\Omega)} \\ &\leq |f|_{L^{\frac{4\theta}{\theta+6}}(\Omega; \mathbb{R}^n)} \left| (|y|^2 - k) \phi_r \right|_{L^{\frac{\theta}{\theta-2}}(\Omega)}^{\frac{3}{4}} + C_1 |f|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)} |y|_{H^1(\Omega; \mathbb{R}^n)}^2 \\ &\leq |f|_{L^{\frac{4\theta}{\theta+6}}(\Omega; \mathbb{R}^n)} \left[\left| (|y|^2 - k) \phi_r \right|_{L^{\frac{\theta}{\theta-2}}(\Omega)} + 1 \right] + C_1 |f|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)} |y|_{H^1(\Omega; \mathbb{R}^n)}^2, \end{aligned} \quad (4.7)$$

here and hereafter C_1 denotes a constant depending only on k, θ, n, m and Ω . Put $L = \sum_{i,j=1}^n |C^{ij}|_{L^{\theta}(\Omega; \mathbb{R}^m)}^2 + \sum_{i=1}^n |D^i|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)} + 1$. By Lemma 3.1, we find that

$$\begin{aligned} &\int_{\Omega} \left(\sum_{i,j=1}^n |C^{ij}|^2 + \sum_{i=1}^n |D^i| + 1 \right) |y|^2 \phi_r dx \\ &\leq C_1 L \left[\int_{\Omega} (|y|^2 \phi_r)^{\frac{\theta}{\theta-2}} dx \right]^{\frac{\theta-2}{\theta}} \leq C_1 L \left| (|y|^2 - k) \phi_r \right|_{L^{\frac{\theta}{\theta-2}}(\Omega)} + C_1 L |\phi_r|_{L^{\frac{\theta}{\theta-2}}(\Omega)} \\ &\leq C_1 L \left| (|y|^2 - k) \phi_r \right|_{L^{\frac{\theta}{\theta-2}}(\Omega)} + C_1 L |y|_{H^1(\Omega; \mathbb{R}^n)}^2. \end{aligned} \quad (4.8)$$

On the other hand, put $u_* = \sqrt{(|y|^2 - k)\phi_r}$. It follows that

$$\begin{aligned}
\int_{\Omega} |\nabla u_*|^2 dx &= \int_{|y|^2 > k} |\nabla u_*|^2 dx = \int_{|y|^2 > k} \left| \frac{2\phi_r y \nabla y + (|y|^2 - k) \nabla \phi_r}{2\sqrt{(|y|^2 - k)\phi_r}} \right|^2 dx \\
&\leq 2 \int_{|y|^2 > k} \left| \frac{\phi_r y \nabla y}{\sqrt{(|y|^2 - k)\phi_r}} \right|^2 dx + \frac{1}{2} \int_{k+r \geq |y|^2 > k} \left| \frac{(|y|^2 - k) \nabla \phi_r}{\sqrt{(|y|^2 - k)\phi_r}} \right|^2 dx \\
&\leq 2 \int_{|y|^2 > k} (|y|^2 - k)^{-1} \phi_r |y|^2 |\nabla y|^2 dx + 2 \int_{k+r \geq |y|^2 > k} (|y|^2 - k) \phi_r^{-1} |y|^2 |\nabla y|^2 dx.
\end{aligned} \tag{4.9}$$

Noting $\phi_r \leq |y|^2 - k$, we see that

$$\begin{aligned}
\int_{|y|^2 > k} (|y|^2 - k)^{-1} \phi_r |y|^2 |\nabla y|^2 dx &= \int_{|y|^2 > k} \phi_r |\nabla y|^2 dx + k \int_{|y|^2 > k} (|y|^2 - k)^{-1} \phi_r |\nabla y|^2 dx \\
&\leq \int_{\Omega} \phi_r |\nabla y|^2 dx + k \int_{|y|^2 > k} |\nabla y|^2 dx \leq \int_{\Omega} \phi_r |\nabla y|^2 dx + k \int_{\Omega} |\nabla y|^2 dx.
\end{aligned} \tag{4.10}$$

Noting that $\phi_r = |y|^2 - k$ whenever $k + r \geq |y|^2$, it is clear that

$$\begin{aligned}
\int_{k+r \geq |y|^2 > k} (|y|^2 - k) \phi_r^{-1} |y|^2 |\nabla y|^2 dx &= \int_{k+r \geq |y|^2 > k} |y|^2 |\nabla y|^2 dx \\
&= \int_{k+r \geq |y|^2 > k} (|y|^2 - k) |\nabla y|^2 dx + k \int_{k+r \geq |y|^2 > k} |\nabla y|^2 dx \\
&= \int_{k+r \geq |y|^2 > k} \phi_r |\nabla y|^2 dx + k \int_{k+r \geq |y|^2 > k} |\nabla y|^2 dx \leq \int_{\Omega} \phi_r |\nabla y|^2 dx + k \int_{\Omega} |\nabla y|^2 dx.
\end{aligned} \tag{4.11}$$

Therefore, by (4.9)–(4.11), we conclude that

$$\int_{\Omega} |\nabla u_*|^2 dx \leq 4 \int_{\Omega} |\nabla y|^2 \phi_r dx + C_1 \int_{\Omega} |\nabla y|^2 dx. \tag{4.12}$$

By (4.12) and Lemma 3.1, for any $0 < \varepsilon_2 < 1$, we end up with

$$\begin{aligned}
|(|y|^2 - k)\phi_r|_{L^{\frac{\theta}{\theta-2}}(\Omega)} &= \left(\int_{\Omega} u_*^{\frac{2\theta}{\theta-2}} dx \right)^{\frac{\theta-2}{\theta}} \leq \varepsilon_2 \int_{\Omega} |\nabla u_*|^2 dx + C_1 \varepsilon_2^{-1} \left(\int_{\Omega} |u_*| dx \right)^2 \\
&\leq 4\varepsilon_2 \int_{\Omega} |\nabla y|^2 \phi_r dx + C_1 \varepsilon_2 |y|_{H^1(\Omega; \mathbb{R}^n)}^2 + C_1 \varepsilon_2^{-1} \left[\int_{\Omega} (|y|^2 - k)^{\frac{1}{2}} \phi_r^{\frac{1}{2}} dx \right]^2 \\
&\leq 4\varepsilon_2 \int_{\Omega} |\nabla y|^2 \phi_r dx + C_1 \varepsilon_2 |y|_{H^1(\Omega; \mathbb{R}^n)}^2 + C_1 \varepsilon_2^{-1} |y|_{L^2(\Omega; \mathbb{R}^n)}^4.
\end{aligned} \tag{4.13}$$

Therefore, substituting (4.13) into (4.7) and (4.8) respectively, we see that

$$\begin{aligned}
\int_{\Omega} |f| |y| \phi_r dx &\leq |f|_{L^{\frac{4\theta}{\theta+6}}(\Omega; \mathbb{R}^n)} \left[4\varepsilon_2 \int_{\Omega} |\nabla y|^2 \phi_r dx + C_1 \varepsilon_2 |y|_{H^1(\Omega; \mathbb{R}^n)}^2 + C_1 \varepsilon_2^{-1} |y|_{L^2(\Omega; \mathbb{R}^n)}^4 + 1 \right] \\
&\quad + C_1 |f|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)} |y|_{H^1(\Omega; \mathbb{R}^n)}^2
\end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} \left(\sum_{i,j=1}^n |C^{ij}|^2 + \sum_{i=1}^n |D^i| + 1 \right) |y|^2 \phi_r dx \\ & \leq 4C_1 L \varepsilon_2 \int_{\Omega} |\nabla y|^2 \phi_r dx + C_1 L (1 + C_1 \varepsilon_2) |y|_{H^1(\Omega; \mathbb{R}^n)}^2 + C_1^2 L \varepsilon_2^{-1} |y|_{L^2(\Omega; \mathbb{R}^n)}^4. \end{aligned}$$

Combining the above inequalities with (4.6) and taking ε_2 sufficiently small such that

$$\left(4|f|_{L^{\frac{4\theta}{\theta+6}}(\Omega; \mathbb{R}^n)} \varepsilon_2 + 4C_1 L \varepsilon_2 \right) C < \frac{1}{2},$$

where C and C_1 are the constants appeared in (4.6) and (4.8) respectively, we arrive at

$$\int_{\Omega} (|\nabla y|^2 \phi_r + |\nabla \phi_r|^2) dx \leq C_2,$$

here and hereafter C_2 is a constant depending on C , C_1 , L , $|f|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)}$ and $|y|_{H^1(\Omega; \mathbb{R}^n)}$, independent of r . Since $\phi_r \in H_0^1(\Omega)$, by the definition of ϕ_r , letting $r \rightarrow +\infty$ in the above inequality, for any fixed $k \geq \operatorname{esssup}_{\Gamma} |y|^2$, we obtain that

$$\int_{\Omega} |\nabla y|^2 (|y|^2 - k)_+ dx + \int_{\Omega} [(|y|^2 - k)_+]^2 dx + \int_{|y|^2 > k} |\nabla (|y|^2)|^2 dx \leq C_2. \quad (4.14)$$

Finally, we construct a sequence of inequalities with respect to A_k , where $A_k = \{x \in \Omega; |y(x)|^2 > k\}$. Again, by (4.6), we get that

$$\begin{aligned} & \int_{\Omega} (|\nabla y|^2 \phi_r + |\nabla \phi_r|^2) dx \\ & \leq C \left[\int_{\Omega} |f| |y| (|y|^2 - k)_+ dx + \int_{\Omega} \sum_{i=1}^n \left(\sum_{j=1}^n |C^{ij}|^2 + |D^i| + 1 \right) |y|^2 (|y|^2 - k)_+ dx \right]. \end{aligned}$$

Letting $r \rightarrow +\infty$ in the above inequality, for any $\varepsilon_3 > 0$, by the Hölder inequality and Lemma 3.1, we see that

$$\begin{aligned} & \int_{A_k} |\nabla y|^2 (|y|^2 - k) dx + \int_{A_k} |\nabla |y|^2|^2 dx \\ & \leq C \int_{A_k} \left[|f| + \sum_{i=1}^n \left(\sum_{j=1}^n |C^{ij}|^2 + |D^i| + 1 \right) \right] (|y|^4 + 1) dx \\ & \leq C (|f|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)} + L) \left(\int_{A_k} (|y|^2 - k)^2_{L^{\frac{2\theta}{\theta-2}}(A_k)} + k^2 |A_k|^{1-\frac{2}{\theta}} \right) + C (|f|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)} + L) |A_k|^{1-\frac{2}{\theta}} \quad (4.15) \\ & \leq C \left(1 + L + |f|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)} \right) \left(\varepsilon_3 |\nabla |y|^2|^2_{L^2(A_k)} + C(\varepsilon_3) \int_{A_k} (|y|^2 - k)^2_{L^2(A_k)} \right) \\ & \quad + C \left(1 + L + |f|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)} \right) k^2 |A_k|^{1-\frac{2}{\theta}}. \end{aligned}$$

Denote $v = |y|^2$ and take ε_3 to be sufficiently small, then by (4.14) and (4.15), one derives that

$$\int_{A_k} |\nabla v|^2 dx \leq C_3 \int_{A_k} |v - k|^2 dx + C_3 k^2 |A_k|^{1-\frac{2}{\theta}}, \quad (4.16)$$

where C_3 denotes a constant only depending only on C , L and $|f|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)}$.

By Lemma 3.2, we take

$$m_0 = 2, \quad l_0 = 2, \quad \sigma = 2, \quad \varepsilon_0 = \frac{2}{m} - \frac{2}{\theta}, \quad \gamma = C_3.$$

Then it follows that

$$\begin{aligned} \operatorname{esssup}_{\Omega} |y| \leq C \left(m, n, \theta, \Omega, \rho, |a^{ij}|_{L^\infty(\Omega)}, L, \left| \frac{\det B^{ij}}{\det B} \right|_{W^{1,\infty}(\Omega)}, \right. \\ \left. |f|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)}, |y|_{L^2(\Omega; \mathbb{R}^n)}, \operatorname{esssup}_{\Gamma} |y| \right). \end{aligned} \quad (4.17)$$

Since y is the weak solution, by Lemma 2.1, we have that

$$|y|_{L^2(\Omega; \mathbb{R}^n)} \leq C(m, n, \theta, \Omega, \rho, |a^{ij}|_{L^\infty(\Omega)}, L) (|f|_{L^{\frac{\theta}{2}}(\Omega; \mathbb{R}^n)} + |g|_{H^1(\Omega; \mathbb{R}^n)}).$$

This, combined with (4.17), yields the desired conclusion in Theorem 2.1. \square

5 Proof of Theorem 2.2

Now, let us prove our second main result, i.e., Theorem 2.2.

Proof of Theorem 2.2. The main idea is the same as that in the proof of Theorem 2.1. First, for any weak solution $y = (y^1, y^2, \dots, y^n)^\top$ to system (1.2), We choose $\varphi = (\varphi^1, \varphi^2, \dots, \varphi^n)^\top \in H_0^1(\Omega; \mathbb{R}^n)$ as a test function, where $\varphi^i = \sum_{l=1}^n E^{il} y^l \zeta_r$, and E^{ij} ($i, j = 1, 2, \dots, n$) are given by assumption **(H)**, while ζ_r is a suitable function to be specified later. By Definition 2.1, it follows that

$$\begin{aligned} & \sum_{i,j=1}^n \sum_{p,q=1}^m \sum_{l=1}^n \int_{\Omega} a_{pq}^{ij} y_{x_p}^j (E^{il} y^l \zeta_r)_{x_q} dx + \sum_{i,j,l=1}^n \int_{\Omega} C^{ij} \cdot \nabla y^j E^{il} y^l \zeta_r dx + \sum_{i,l=1}^n \int_{\Omega} D^i \cdot y E^{il} y^l \zeta_r dx \\ &= \sum_{i,l=1}^n \int_{\Omega} f^i E^{il} y^l \zeta_r dx. \end{aligned}$$

This implies that

$$\begin{aligned} & \sum_{i,j=1}^n \sum_{p,q=1}^m \sum_{l=1}^n \int_{\Omega} \left[a_{pq}^{ij} E^{il} y_{x_p}^j y_{x_q}^l \zeta_r + a_{pq}^{ij} E^{il} y_{x_p}^j y^l (\zeta_r)_{x_q} + a_{pq}^{ij} (E^{il})_{x_q} y_{x_p}^j y^l \zeta_r \right] dx \\ &+ \sum_{i,j,l=1}^n \int_{\Omega} C^{ij} \cdot \nabla y^j E^{il} y^l \zeta_r dx + \sum_{i,l=1}^n \int_{\Omega} D^i \cdot y E^{il} y^l \zeta_r dx \\ &= \sum_{i,l=1}^n \int_{\Omega} f^i E^{il} y^l \zeta_r dx. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sum_{i,j=1}^n \sum_{p,q=1}^m \sum_{l=1}^n \int_{\Omega} \left[a_{pq}^{ij} E^{il} y_{x_p}^j y_{x_q}^l \zeta_r + a_{pq}^{ij} E^{il} y_{x_p}^j y^l (\zeta_r)_{x_q} \right] dx \\ & \leq C_4 \int_{\Omega} \left[|f| |y| \zeta_r + \left(1 + \sum_{i,j=1}^n |C^{ij}| \right) |\nabla y| |y| \zeta_r + \sum_{i=1}^n |D^i| |y|^2 \zeta_r \right] dx, \end{aligned} \quad (5.1)$$

here and hereafter C_4 denotes a constant depending only on $n, m, \rho, |a_{pq}^{ij}|_{L^\infty(\Omega)}$ and $|E^{ij}|_{W^{1,\infty}(\Omega)}$ ($i, j = 1, \dots, n; p, q = 1, \dots, m$).

Next, we estimate the terms in the left side of (5.1). For this purpose, by (2.8), condition 2) in assumption **(H)** and the Cramer rule, we see that for any $p, q = 1, \dots, m$, functions E^{ij} ($i, j = 1, \dots, n$) (given by assumption **(H)**) satisfy $\sum_{l=1}^n a_{pq}^{li} E^{lj} = f_{pq} h^{ij}$. In particular, by $h^{11} = 1$, we see that $f_{pq} = \sum_{l=1}^n a_{pq}^{l1} E^{l1}$. Therefore,

$$\sum_{l=1}^n a_{pq}^{li} E^{lj} = h^{ij} \sum_{l=1}^n a_{pq}^{l1} E^{l1}. \quad (5.2)$$

Hence,

$$\begin{aligned} & \sum_{i,j=1}^n \sum_{p,q=1}^m \sum_{l=1}^n \int_{\Omega} a_{pq}^{ij} E^{il} y_{x_p}^j y^l (\zeta_r)_{x_q} dx = \sum_{p,q=1}^m \sum_{j,l=1}^n \int_{\Omega} \left(\sum_{i=1}^n a_{pq}^{i1} E^{i1} \right) h^{jl} y_{x_p}^j y^l (\zeta_r)_{x_q} dx \\ & = \sum_{p,q=1}^m \int_{\Omega} \left(\sum_{i=1}^n a_{pq}^{i1} E^{i1} \right) \left[\frac{1}{2} \sum_{j=1}^n h^{jj} (y^j)^2 + \sum_{j,l \in \{1,2,\dots,n\}, j < l} h^{jl} y^j y^l \right] (\zeta_r)_{x_q} dx \\ & \quad - \sum_{p,q=1}^m \int_{\Omega} \frac{1}{2} \left(\sum_{i=1}^n a_{pq}^{i1} E^{i1} \right) \left[\sum_{j,l=1}^n (h^{jl})_{x_p} y^j y^l \right] (\zeta_r)_{x_q} dx. \end{aligned} \quad (5.3)$$

On the other hand, by condition 4) in assumption **(H)** and noting (5.2), it is easy to see that

$$M = \begin{pmatrix} \sum_{l=1}^n a_{11}^{l1} E^{l1} & \cdots & \sum_{l=1}^n a_{1m}^{l1} E^{l1} & \cdots & \sum_{l=1}^n a_{11}^{l1} E^{ln} & \cdots & \sum_{l=1}^n a_{1m}^{l1} E^{ln} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{l=1}^n a_{m1}^{l1} E^{l1} & \cdots & \sum_{l=1}^n a_{mm}^{l1} E^{l1} & \cdots & \sum_{l=1}^n a_{m1}^{l1} E^{ln} & \cdots & \sum_{l=1}^n a_{mm}^{l1} E^{ln} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{l=1}^n a_{11}^{ln} E^{l1} & \cdots & \sum_{l=1}^n a_{1m}^{ln} E^{l1} & \cdots & \sum_{l=1}^n a_{11}^{ln} E^{ln} & \cdots & \sum_{l=1}^n a_{1m}^{ln} E^{ln} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{l=1}^n a_{m1}^{ln} E^{l1} & \cdots & \sum_{l=1}^n a_{mm}^{ln} E^{l1} & \cdots & \sum_{l=1}^n a_{m1}^{ln} E^{ln} & \cdots & \sum_{l=1}^n a_{mm}^{ln} E^{ln} \end{pmatrix}_{nm \times nm}.$$

Therefore,

$$\sum_{i,j=1}^n \sum_{p,q=1}^m \sum_{l=1}^n \int_{\Omega} a_{pq}^{ij} E^{il} y_{x_p}^j y_{x_q}^l \zeta_r dx \geq \rho_3 \int_{\Omega} |\nabla y|^2 \zeta_r dx. \quad (5.4)$$

Combining (5.3) and (5.4) with (5.1), we have

$$\begin{aligned} & \int_{\Omega} \left\{ |\nabla y|^2 \zeta_r + \sum_{p,q=1}^m \left(\sum_{i=1}^n a_{pq}^{i1} E^{i1} \right) \psi_{x_p} (\zeta_r)_{x_q} \right\} dx \\ & \leq C_5 \int_{\Omega} \left[|f| |y| \zeta_r + \left(1 + \sum_{i,j=1}^n |C^{ij}| \right) |\nabla y| |y| \zeta_r + \sum_{i=1}^n |D^i| |y|^2 \zeta_r + |y|^2 |\nabla \zeta_r| \right] dx, \end{aligned} \quad (5.5)$$

where $\psi = \sum_{j,l=1}^n h^{jl} y^j y^l$ and C_5 depends only on C_4 , ρ_3 and $|h^{ij}|_{W^{1,\infty}(\Omega)}$ ($i, j = 1, 2, \dots, n$).

For $s, r > 0$ and $k > \sup_{\Gamma} \psi^s$, denote

$$A_k = \{x \in \Omega \mid \psi^s(x) > k\}, \quad A_k^r = \{x \in \Omega \mid k < \psi^s(x) < k + r\}.$$

Moreover, we choose $\zeta_r = \min\{r, (\psi^s - k)_+\}$. Then, by (5.5), and using condition 3) in assumption **(H)**, we conclude that

$$\begin{aligned} & \int_{A_k} |\nabla y|^2 \zeta_r dx + \int_{A_k^r} \psi^{s-1} |\nabla \psi|^2 dx \\ & \leq C_6 \int_{A_k} \left[|f| |y| \zeta_r + \left(1 + \sum_{i,j=1}^n |C^{ij}|^2 + \sum_{i=1}^n |D^i| \right) |y|^2 \zeta_r + |y|^2 |\nabla \zeta_r| \right] dx, \end{aligned} \quad (5.6)$$

where C_6 denotes a constant depending only on s , C_5 and ρ_2 . Moreover, using condition 1) in assumption **(H)**, it follows that

$$\psi \geq \rho_1 |y|^2.$$

Now, let us estimate the right side of (5.6). First, by Hölder's inequality, for any $\varepsilon_4 > 0$, we have that

$$\int_{A_k} |f| |y| \zeta_r dx \leq C_7 \int_{A_k} |y| \zeta_r dx \leq \varepsilon_4 \int_{A_k} \zeta_r^{\frac{s+1}{s}} dx + \varepsilon_4^{-1} C_7 \int_{A_k} |y|^{s+1} dx, \quad (5.7)$$

here and hereafter C_7 denotes a constant depending only on C_6 , Ω , ρ_1 , $|f|_{L^\infty(\Omega; \mathbb{R}^n)}$, $\sum_{i,j=1}^n |C^{ij}|_{L^\infty(\Omega; \mathbb{R}^m)}$

and $\sum_{i=1}^n |D^i|_{L^\infty(\Omega; \mathbb{R}^n)}$. Next,

$$\begin{aligned} & \int_{A_k} \left(1 + \sum_{i,j=1}^n |C^{ij}|^2 + \sum_{i=1}^n |D^i| \right) |y|^2 \zeta_r dx \leq C_7 \int_{A_k} |y|^2 \zeta_r dx \\ & \leq \varepsilon_4 \int_{A_k} \zeta_r^{\frac{s+1}{s}} dx + \varepsilon_4^{-1} C_7 \int_{A_k} |y|^{2s+2} dx. \end{aligned} \quad (5.8)$$

Further,

$$\begin{aligned} \int_{A_k} |y|^2 |\nabla \zeta_r| dx &\leq C_7 \int_{A_k^r} |y|^2 \psi^{s-1} |\nabla \psi| dx \leq \varepsilon_4 \int_{A_k^r} \psi^{s-1} |\nabla \psi|^2 dx + \varepsilon_4^{-1} C_7 \int_{A_k^r} |y|^4 \psi^{s-1} dx \\ &\leq \varepsilon_4 \int_{A_k^r} \psi^{s-1} |\nabla \psi|^2 dx + \varepsilon_4^{-1} C_7 \int_{A_k^r} |y|^{2s+2} dx. \end{aligned} \quad (5.9)$$

On the other hand, by Poincaré's inequality,

$$\int_{\Omega} \zeta_r^{\frac{s+1}{s}} dx \leq C_7 \int_{\Omega} \left| \nabla (\zeta_r^{\frac{s+1}{2s}}) \right|^2 dx \leq C_7 \int_{A_k^r} \zeta_r^{\frac{1-s}{s}} \psi^{2s-2} |\nabla \psi|^2 dx \leq C_7 \int_{A_k^r} \psi^{s-1} |\nabla \psi|^2 dx. \quad (5.10)$$

Therefore, by (5.6)–(5.10), taking ε_4 sufficiently small, we get that

$$\int_{A_k} |\nabla y|^2 \zeta_r dx + \int_{A_k^r} |\nabla (\psi^{\frac{s+1}{2}})|^2 dx \leq C_7 \int_{A_k} (|y|^{2s+2} + |y|^{s+1}) dx.$$

Letting $r \rightarrow +\infty$ in the above inequality, we have that

$$\int_{A_k} |\nabla y|^2 (\psi^s - k) dx + \int_{A_k} |\nabla (\psi^{\frac{s+1}{2}})|^2 dx \leq C_7 \int_{A_k} (|y|^{2s+2} + |y|^{s+1}) dx. \quad (5.11)$$

Notice that for any given constant $s \leq \frac{2}{m-2}$ (if $m \leq 2$, s can be any positive number), the right side of (5.11) is finite.

Denote $\tilde{k} = k^{\frac{s+1}{2s}}$ and $A_{\tilde{k}} = \{x \in \Omega \mid \psi^{\frac{s+1}{2}} > \tilde{k}\} = A_k$. Then by (5.11), it follows that

$$\begin{aligned} \int_{A_{\tilde{k}}} |\nabla (\psi^{\frac{s+1}{2}})|^2 dx &\leq C_7 \int_{A_{\tilde{k}}} \psi^{s+1} dx + C_7 \int_{A_{\tilde{k}}} \psi^{\frac{s+1}{2}} dx \\ &\leq C_7 \int_{A_{\tilde{k}}} (\psi^{\frac{s+1}{2}} - \tilde{k})^2 dx + C_7 \tilde{k}^2 |A_{\tilde{k}}| + C_7 \int_{A_{\tilde{k}}} (\psi^{\frac{s+1}{2}} - \tilde{k}) dx + C_7 \tilde{k} |A_{\tilde{k}}| \\ &\leq C_7 \int_{A_{\tilde{k}}} (\psi^{\frac{s+1}{2}} - \tilde{k})^2 dx + C_7 \tilde{k}^2 |A_{\tilde{k}}|. \end{aligned} \quad (5.12)$$

By Lemma 3.2, we take

$$u = \psi^{\frac{s+1}{2}}, \quad m_0 = \sigma = l_0 = 2, \quad k = \tilde{k}, \quad \gamma = C_7, \quad \varepsilon_0 = \frac{2}{m}.$$

Then, using also Lemma 2.1, it follows that

$$\begin{aligned} \text{esssup}_{\Omega} |y| &\leq C(m, n, \Omega, \rho, \rho_1, \rho_2, \rho_3, |a_{pq}^{ij}|_{L^\infty(\Omega)}, |C^{ij}|_{L^\infty(\Omega; \mathbb{R}^m)}, |D^i|_{L^\infty(\Omega; \mathbb{R}^n)}, \\ &\quad |E^{ij}|_{W^{1,\infty}(\Omega)}, |h^{ij}|_{W^{1,\infty}(\Omega)}, |g|_{H^1(\Omega; \mathbb{R}^n)}, |f|_{L^\infty(\Omega; \mathbb{R}^n)}, \text{esssup}_{\Gamma} |y|). \end{aligned}$$

This completes the proof of Theorem 2.2. □

6 An example

In this section we give an example, in which the coefficients a_{pq}^{ij} ($i, j = 1, 2, \dots, n; p, q = 1, 2, \dots, m$) of system (1.2) satisfy all of the assumptions in Theorem 2.2.

For any given functions $b^{ij} \in W^{1,\infty}(\Omega)$ and $g_{pq} \in L^\infty(\Omega)$ ($i, j = 1, 2, \dots, n; p, q = 1, 2, \dots, m$) such that $g_{pq} > 0$ and the following matrix is uniformly positive definite:

$$G := \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ g_{21} & g_{22} & \cdots & g_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mm} \end{pmatrix},$$

we take

$$a_{pq}^{ij} = b^{ij} g_{pq}, \quad h^{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad f_{pq} = \frac{L_{pq}}{(g_{pq})^{n-1}}.$$

Then it is easy to check the following assertions:

- i) Condition 1) in assumption **(H)** holds, since $V = I_{n \times n}$;
- ii) By the definition of a_{pq}^{ij} and f_{pq} , and $b^{ij} \in W^{1,\infty}(\Omega)$ ($i, j = 1, 2, \dots, n; p, q = 1, 2, \dots, m$), if (2.8) holds, condition 2) in assumption **(H)** is satisfied;
- iii) If $b^{11} \neq 0$, $b^{i1} = 0$, $i = 2, 3, \dots, n$, and $E^{11} := \frac{f_{pq}}{L_{pq}} \sum_{l=1}^n h^{l1} v_{pq}^{l1} > 0$ in Ω , then by (2.5), conditions 3) and 4) in assumption **(H)** hold true. Notice that by the definition of E^{11} , $E^{11} > 0$ if and only if

$$\det \begin{pmatrix} b^{22} & b^{23} & \cdots & b^{2n} \\ b^{32} & b^{33} & \cdots & b^{3n} \\ \vdots & \vdots & \vdots & \vdots \\ b^{n2} & b^{n3} & \cdots & b^{nn} \end{pmatrix} > 0. \quad (6.1)$$

Moreover, under condition (6.1), the hypothesis (2.8) also holds;

- iv) Condition (2.5) is equivalent to that the following matrix is uniformly positive definite:

$$K := \begin{pmatrix} b^{11}G & \frac{1}{2}b^{12}G & \cdots & \frac{1}{2}b^{1n}G \\ \frac{1}{2}b^{12}G & b^{22}G & \cdots & \frac{1}{2}(b^{2n} + b^{n2})G \\ \frac{1}{2}b^{13}G & \frac{1}{2}(b^{32} + b^{23})G & \cdots & \frac{1}{2}(b^{3n} + b^{n3})G \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2}b^{1n}G & \frac{1}{2}(b^{2n} + b^{n2})G & \cdots & b^{nn}G \end{pmatrix}_{nm \times nm}.$$

Notice that if for some constant $\rho_* > 0$,

$$b^{ii} \geq \rho_* \quad \text{and} \quad b^{ij} \leq n\rho_* \quad (i, j = 1, 2, \dots, n; i \neq j), \quad (6.2)$$

then the matrix K is uniformly positive definite.

By the above assertions i)–iv), suppose that the coefficients a_{pq}^{ij} ($i, j = 1, 2, \dots, n$; $p, q = 1, 2, \dots, m$) of system (1.2) satisfy that

$$a_{pq}^{ij} = b^{ij} g_{pq},$$

where $b^{ij} \in W^{1,\infty}(\Omega)$, $g_{pq} \in L^\infty(\Omega)$, $b^{11} \neq 0$, $b^{i1} = 0$ ($i = 2, \dots, n$), $g_{pq} > 0$, and G is uniformly positive definite, and (6.1)–(6.2) are satisfied. Then, by Theorem 2.2, we conclude the boundedness of weak solutions to the corresponding system (1.2).

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